where $C_k$ are arbitrary constants. Substituting into the integral equation (2.17) for $n=0$ the value of $\tau_0(\psi)$ defined by (3.7), taking into account that $f_0'(\tau)=0$ and that the integral on the left is some constant for any $\varphi \in [-\theta, \theta]$, in particular for $\varphi=0$, we have the condition for obtaining the constant $C_0$. From this we have

$$C_0 = \pi^3 \left\{ \int_{-\theta}^{\theta} \frac{\operatorname{cn}(K\psi/\pi)\, k(\psi)\, d\psi}{\sqrt{\operatorname{sn}^2(K\theta/\pi) - \operatorname{sn}^2(K\psi/\pi)}} \right\}^{-1} \tag{3.8}$$

where $k(\psi)$ is defined by Eq.(2.18).

The constants $C_k$ $(k=1,2,\ldots)$ can be obtained using the first condition (2.16).

Note that when the functions $\varphi_k(\xi)$ and $g_k(x)$ are odd, the integral equations (2.17) can also be reduced to the singular equation (3.5) using the second of formulas (3.3).

The method of homogeneous solutions, described in Sect.2, can also be used to investigate the contact problem of the pressing of a stamp into a cylindrical surface of a ring sector, when the surfaces $|\varphi| = \gamma$ are free of tangential stresses and normal displacements.

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Translated by J.J.D.

*PMM U.S.S.R.*,Vol.47,No.5,pp.645-650,1983
Printed in Great Britain

0021-8928/83 $10.00+0.00

UDC 539.3.534.1

# VIBRATION OF A CYLINDER ON AN ELASTIC LAYER PARTLY FIXED TO A RIGID BASE*

S.P. PEL'TS

The problem of non-resonant harmonic oscillations of an elastic cylinder on an elastic layer is considered. The contact between the cylinder and the layer is over a circular region $\Omega_1$ of radius $R_1$ without friction. The layer rests on a rigid base. At the layer-base interface there are two types of contact: in the circular region $\Omega_2$ of radius $R_2$ there is rigid adherence, while outside it there is no friction. The length of the projection of the distance between the centres of regions $\Omega_1$ and $\Omega_2$ on the horizontal plane is $d$. Problems of this kind are encountered in flaw detection in foundations and adhesive joints.

Problems of the vibration of a rigid body (stamp) on the surface of an elastic layer under various contact conditions were considered in /1/. Here the stamp is replaced by an elastic cylinder, which leads to a qualitatively new mechanical system that takes into account the effect of the finite elastic body. A many-sided analysis of the cylinder harmonic oscillations is given in /2/.

1. We combine the cylinder axis with the $\zeta$ axis and locate the origin of coordinates on the upper face of the layer. All quantities relating to the cylinder will be denoted by the subscript 1, and those relating to the layer by the subscript 2; $\lambda_n, \mu_n, \rho_n$ $(n=1,2)$ are the

*Prikl.Matem.Mekhan.,47,5,799-804,1983

Lame coefficients and the density of the material, $H$ is the cylinder height and $h$ the layer thickness. On the upper end surface of the cylinder there are no tangential stresses and the displacements vary as $w^{(1)}(\rho, H, t) = w^\circ e^{-i\omega t}$. The lateral surface of the cylinder is free of stresses. We will change to dimensionless quantities

$$r = h^{-1}\rho, \quad z = h^{-1}\zeta, \quad h_1 = h^{-1}H, \quad w^* = h^{-1}w^\circ, \quad b = h^{-1}d, \quad a_n = h^{-1}R_n \quad (n = 1,2)$$

The solution for the cylinder is constructed in the form of a series in the homogeneous solutions of the problem of the vibration of an infinite cylinder with a free lateral surface /3/. We expand the displacements and stresses in Fourier series in the angular coordinate $\varphi$, and ignoring the time factor, obtain

$$u^{(1)}(r, z, \varphi) = \sum_{k=-\infty}^{\infty} \sum_{s=1}^{\infty} C_s^k U_s(r, k) \frac{\cos n_{ks}(h_1 - z)}{\cos n_{ks} h_1} e^{ik\varphi} \tag{1.1}$$

$$v^{(1)}(r, z, \varphi) = \sum_{k=-\infty}^{\infty} \sum_{s=1}^{\infty} C_s^k V_s(r, k) \frac{\cos n_{ks}(h_1 - z)}{\cos n_{ks} h_1} e^{ik\varphi}$$

$$w^{(1)}(r, z, \varphi) = \sum_{k=-\infty}^{\infty} \sum_{s=1}^{\infty} C_s^k W_s(r, k) \frac{\sin n_{ks}(h_1 - z)}{\cos n_{ks} h_1} e^{ik\varphi} + w^*$$

where $C_s^k$ are constants to be determined, $n_{ks}$ are the roots of the dispersion equation for an infinite cylinder, and $U_s(r, k)$, $V_s(r, k)$, $W_s(r, k)$ are known functions, which owing to their complexity are not given here. The expressions obtained for the displacements (1.1) enable us to satisfy the boundary conditions on the top face of the cylinder.

2. Let us obtain a solution of the layer problem. Ignoring the term $e^{-i\omega t}$ we write the boundary conditions as

$$\sigma_z^{(1)}(r, 0, \varphi) = \sigma_z^{(2)}(r, 0, \varphi), \quad \tau_{zr}^{(1)}(r, 0, \varphi) = \tau_{z\varphi}^{(1)}(r, 0, \varphi) = 0 \tag{2.1}$$

$$(r, \varphi \in \Omega_1)$$

$$u^{(2)}(r, -1, \varphi) = v^{(2)}(r, -1, \varphi) = 0 \quad (r, \varphi \in \Omega_2)$$

$$\tau_{zr}^{(2)}(r, -1, \varphi) = \tau_{z\varphi}^{(2)}(r, -1, \varphi) = 0 \quad (r, \varphi \overline{\in} \Omega_2)$$

$$w^{(2)}(r, -1, \varphi) = 0 \quad (0 \leqslant r < \infty, 0 \leqslant \varphi < 2\pi)$$

All the components of the stress and displacement tensors can be represented in the form of Fourier series

$$\tau_k^{(2)}(r, s, \varphi) = \sum_{m=-\infty}^{\infty} \tau_{km}^{(2)}(r, s) e^{im\varphi}, \quad u_k^{(2)}(r, s, \varphi) = \sum_{m=-\infty}^{\infty} u_{km}^{(2)}(r, s) e^{im\varphi} \quad (s = -1, 0; \ k = 1, 2, 3)$$

$$\tau_1^{(2)} \equiv \tau_{zr}^{(2)}, \quad \tau_2^{(2)} \equiv \tau_{z\varphi}^{(2)}, \quad \tau_3^{(2)} \equiv \sigma_z^{(2)}, \quad u_1^{(2)} \equiv u^{(2)}, \quad u_2^{(2)} \equiv v^{(2)}, \quad u_3^{(2)} \equiv w^{(2)}$$

Applying the two-dimensional Fourier integral transform to the Lamé equations and using the boundary conditions (2.1), we obtain formulas defining the displacements of the points of the layer. Then equating the values obtained for the displacements on the lower face, of the layer surface to the known boundary conditions in the region $\Omega_2$, we obtain an integral equation of the first kind for the contact stresses in $\Omega_2$

$$\int_0^{a_2}\int_\sigma S_m(\alpha r) K(\alpha) S_m(\alpha\rho) q_m(n, p, s, \rho) \rho\alpha \, d\alpha \, d\rho = f_m(n, p, s, r) \tag{2.2}$$

$$(0 \leqslant r \leqslant a_2; \ n = 1, 2; \ s = 1, 2, \ldots; \ m, \ p = 0, \pm 1, \pm 2, \ldots)$$

$$q_m(r) = \{q_{1m}(r), \ q_{2m}(r)\} = \sum_{p=-\infty}^{\infty} \sum_{s=1}^{\infty} \sum_{n=1}^{2} (-1)^{m+p} C_s^p b_{np}^s q_m$$

$$(n, p, s, r)$$

$$q_{1m}(r) = \tau_{1m}^{(2)}(r, -1) + i\tau_{2m}^{(2)}(r, -1), \quad q_{2m}(r) = \tau_{1m}^{(2)}(r, -1) - i\tau_{2m}^{(2)}(r, -1)$$

$$f_m(n, p, s, r) = \{f_{m-1}(n, p, s, r), \ f_{m+1}(n, p, s, r)\}$$

where $b_{np}^s$ are known constants, $S_m(\alpha r)$ is a diagonal matrix with elements $J_{m-1}(\alpha r)$, $J_{m+1}(\alpha r)$, $J_m(z)$ are Bessel functions, and $H_m^{(n)}(z)$ are Hankel functions. The elements of the matrix $K(\alpha)$ are defined by the relations

$$k_{11}(\alpha) = k_{22}(\alpha) = N_1(\alpha) + N_2(\alpha),\quad k_{12}(\alpha) = k_{21}(\alpha) = N_1(\alpha) - N_2(\alpha)$$

$$N_1(\alpha)\,\Delta(\alpha) = s_2(4\alpha^4 + b^2{}_3)\operatorname{ch} s_1 \operatorname{ch} s_2 - \alpha^2(4s_1s_2^2 + s_1^{-1}b_3^2) \times \operatorname{sh} s_1 \operatorname{sh} s_2 - 4\alpha^2 s_2 b_3$$

$$N_2(\alpha) = s_2^{-1}\operatorname{cth} s_2,\ \Delta(\alpha) = b_2^2[b_3^2 \operatorname{ch} s_1 \operatorname{sh} s_2 - 4s_1s_2\,\alpha^2 \times \operatorname{sh} s_1 \operatorname{ch} s_2]$$

$$s_k^2 = \alpha^2 - b_k^2\ (k = 1, 2);\ b_1^2 = \rho_2(\omega h)^2(\lambda_2 + 2\mu_2)^{-1},\ b_2^2 =$$
$$= \rho_2(\omega h)^2\mu_2^{-1},\ b_3 = 2\alpha^2 - b_2^2$$

$$f_{m\pm1}(n, p, s, r) = \pm\int_\sigma \Phi_m(n, p, s, \alpha)\, J_{m\pm1}(\alpha r)\,\alpha d\alpha$$

$$\Phi_m(n, p, s, \alpha) = \alpha R(\alpha)\, J_{m-p}(\alpha b)\, M_p(\alpha, \sigma_{np}^s)$$

$$M_p(\alpha, x) = a_1\frac{\alpha J_p(xa_1)\,J_{p-1}(\alpha a_1) - xJ_{p-1}(xa_1)\,J_p(\alpha a_1)}{(x^2 - \alpha^2)\,J_p(\sigma_{1p}^s a_1)}$$

$$\sigma_{jp}^s = r_j^2 - n_{ps}^2,\ r_1^2 = \rho_1(\omega h)^2(\lambda_1 + 2\mu_1)^{-1},\ r_2^2 = \rho_1(\omega h)^2\mu_1^{-1}$$

$$R(\alpha) = b_2^2[b_3 \operatorname{sh} s_2 - 2s_1s_2 \operatorname{sh} s_1]\,\Delta^{-1}(\alpha)$$

As $\alpha \to \infty$ the elements of matrix $K(\alpha)$ behave as $O(|\alpha|^{-1})$. The unique solvability of Eq. (2.2) was established in /4/ in the case of asymptotic forms of this kind.

A method of solving (2.2) was devised in /1/. The disposition of the contour $\sigma$ is dictated by the principle of limit absorption (damping). Following /1/ we obtain

$$q_m(n, p, s, r) = \int_\sigma S_m(\alpha r)\, K^{-1}(\alpha)\, A_m(n, p, s, \alpha)\,\alpha d\alpha + \tag{2.3}$$
$$\frac{1}{4\pi^2}\int_\Gamma \varkappa_{2m}(\alpha)\, S_m(\alpha r)\, K_+^{-1}(\alpha)\, X_m(n, p, s, \alpha)\, d\alpha$$
$$A_m(n, p, s, \alpha) = \{1, -1\}\,\Phi_m(n, p, s, \alpha)$$

We assume that the factorization $K(\alpha) = K_-(\alpha)\,K_+(\alpha)$ is carried out. The vector function $X_m(n, p, s, \alpha)$ is determined from the equation of the second kind

$$X_m(n, p, s, z) + \frac{1}{4\pi^2}\int_\Gamma\int_{\Gamma_-} K_+(\alpha)\, D_m(\alpha, u)\, K_+^{-1}(u) \times \tag{2.4}$$
$$X_m(n, p, s, u)\,\frac{du\,d\alpha}{\alpha - z} = \int_\Gamma\int_\sigma \varkappa_{1m}(\alpha)\, K_+(\alpha)\,\theta(m, \alpha, u, a_2) \times$$
$$K^{-1}(u)\, A_m(n, p, s, u)\,\frac{u\,du\,d\alpha}{\alpha - z}\quad (z > \Gamma > \Gamma_-)$$

The contour $\Gamma_-$ lies below $\Gamma$, but such that between them the integrands are regular. The following notation is used here:

$$\varkappa_{1m}(\alpha) = \frac{\sqrt{a_2}}{iH_m^{(2)}(\alpha a_2)},\quad \varkappa_{2m}(\alpha) = \pi\alpha\sqrt{a_2}\,H_m^{(2)}(\alpha a_2)$$

$\theta(m, \alpha, u, a)$ is a second-order diagonal matrix with elements

$$\theta_{11} = [\alpha J_{m-1}(ua)\ H_m^{(2)}(\alpha a) - uJ_m(ua)H_{m-1}^{(2)}(\alpha a)]\,(\alpha^2 - u^2)^{-1}$$
$$\theta_{22} = [uJ_m(ua)H_{m+1}^{(2)}(\alpha a) - \alpha J_{m+1}(ua)H_m^{(2)}(\alpha a)](\alpha^2 - u^2)^{-1}$$
$$D_m(\alpha, u) = \varkappa_{1m}(\alpha)\,\theta(m, \alpha, u, a_2)\varkappa_{2m}(\alpha) - (\alpha - u)^{-1} E$$

where $E$ is the unit matrix. Equation (2.4) is effectively solved by simultaneous deformation of the contours $\Gamma$ and $\Gamma_-$ into the lower half plane. Calculating the residues of the integrands at the poles intersected by the deformed contours, and neglecting small integral terms, we reduce the solution of the equation to the solution of a finite set of linear algebraic equations. Approximate factorization of the matrix-function $K(\alpha)$ is carried out on by approximating it by a matrix with rational-fractional elements. The error of the approximate solution is estimated using Theorem 2 of /4/.

The displacements of the points of the upper face of the layer are given by the formula

$$w_m^{(2)}(r) = \varepsilon\sum_{n=1}^{2}\Big\langle\int_\sigma\sum_{s=1}^{\infty} C_s^m b_{nm}^s P_1(\alpha)\, M_m(\alpha, \sigma_{nm}^s) \times \tag{2.5}$$
$$J_m(\alpha r)\,\alpha d\alpha + \int_0^{a_1}\int_\sigma \sum_{l=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{s=1}^{\infty}(-1)^{l+p}\, C_s^p b_{np}^s P_2(u) \times$$

$$J_m(ur)\,J_{m-l}(ub)[J_{l+1}(u\rho)\,q_{1l}(n,p,s,\rho)-$$
$$J_{l-1}(u\rho)\,q_{2l}(n,p,s,\rho)]\,u\rho du\,d\rho\Big\rangle\quad(0\leqslant r<\infty),\quad \varepsilon=\mu_1/\mu_2$$
$$P_1(\alpha)\,\Delta(\alpha)=-b_2{}^4 s_1\,\mathrm{sh}\,s_1\,\mathrm{sh}\,s_2,\quad P_2(\alpha)\,\Delta(\alpha)=-\alpha b_2{}^2\times$$
$$[s_1 s_2\,\mathrm{sh}\,s_1-0{,}5\,b_2\,\mathrm{sh}\,s_2]$$

where the first integral term takes into account the effect of the cylinder, and the second the presence of rigid adherence at the lower face of the layer. Let us represent the first term in the form of a series in residues

$$w_{m,1}^{(2)}(r)=\varepsilon\sum_{n=1}^{2}\sum_{s=1}^{\infty}C_s^m\Big[\sum_{k=1}^{\infty}A_{ks}^{mn}J_m(\gamma_k r)\;+B_{sn}^mJ_m(\sigma_{nm}^s r)\Big]\quad(0\leqslant r\leqslant a_1)\tag{2.6}$$

$$w_{m,1}^{(2)}(r)=\varepsilon\sum_{n=1}^{2}\sum_{s=1}^{\infty}C_s^m\sum_{k=1}^{\infty}D_{ks}^{mn}H_m^{(2)}(\gamma_k r)\quad(r>a_1),\qquad\Delta(\gamma_k)=0\tag{2.7}$$

where $A_{ks}{}^{mn}, B_{sn}{}^{m}, D_{ks}{}^{mn}$ are constants. The number of standing waves under the cylinder and their characteristics therefore depend on all the parameters of the layer and the cylinder. However, the number of waves propagating from the cylinder is equal to the number of real roots $\gamma_k$, and their phase velocities depend only on the layer parameters. The effect of the cylinder characteristics appears in their amplitude and phase shift. The problem has thus been reduced to determining the constants $C_s^k$.

3. Formulas of generalized orthogonality were obtained in /5/ for the dynamically homogeneous solutions of a cylinder, which in a cylindrical coordinate system have the form

$$(U_j(r,-k),\tau_{1p}^{(1)}(r,k))+\delta_k(V_j(r,-k),\tau_{2p}^{(1)}(r,k))\;-(\sigma_{zj}^{(1)}(r,k),W_p(r,-k))=\begin{cases}0, & n_{kj}^2\neq n_{kp}^2\\ R_j^{\,k}, & n_{kj}^2=n_{kp}^2\end{cases}\tag{3.1}$$

$$(k=0,\pm1,\pm2,\ldots),\quad \delta_k=\begin{cases}1, & k\neq0\\0, & k=0\end{cases}$$

$$(f(r,-k),g(r,k))=\int_0^{a_1}f(r,-k)\,g(r,k)\,rdr$$

The stresses $\tau_{1p}{}^{(1)}, \tau_{2p}{}^{(1)}, \sigma_{zp}{}^{(1)}$ are calculated using (1.1). We will satisfy the boundary conditions on the lower face of the cylinder

$$w^{(1)}(r,0,\varphi)=w^{(2)}(r,0,\varphi),\quad \tau_{zr}{}^{(1)}(r,0,\varphi)=\tau_{z\varphi}{}^{(1)}(r,0,\varphi)=0$$
$$(r,\varphi\in\Omega_1)$$

As a result we obtain the following set of equations

$$(1-\delta_k)\,w^*-\sum_{s=1}^{\infty}C_s{}^kW_s(r,k)\,\mathrm{tg}\,n_{ks}h_1=\varepsilon\sum_{s=1}^{\infty}C_s{}^kL_1(k,s,r)+\varepsilon\sum_{p=-\infty}^{\infty}\sum_{s=1}^{\infty}C_s{}^pL_2\quad(k,s,p,r)\tag{3.2}$$

$$\sum_{s=1}^{\infty}C_s{}^k\tau_{1s}^{(1)}(r,k)\,\mathrm{tg}\,n_{ks}h_1=0$$
$$\delta_k\sum_{s=1}^{\infty}C_s{}^k\tau_{2s}^{(1)}(r,k)\,\mathrm{tg}\,n_{ks}h_1=0$$
$$(0\leqslant r\leqslant a_1,\ k=0,\pm1,\pm2,\ldots)$$

where $L_1(k,s,r)$, $L_2(k,s,p,r)$ are known functions. Applying to (3.2) the relation of generalized orthogonality, we obtain

$$C_l{}^k=(1-\delta_k)\,A_l{}^k+\varepsilon\sum_{s=1}^{\infty}C_s{}^kF_1(s,l,k)+\;\varepsilon\sum_{p=-\infty}^{\infty}\sum_{s=1}^{\infty}C_s{}^pF_2(s,p,l,k)\quad(l=1,2,\ldots;k=0,\pm1,\pm2,\ldots)\tag{3.3}$$

$$F_1(s,l,k)\,R_l{}^k=-\sum_{n=1}^{2}\sum_{j=1}^{2}b_{nk}^s b_{jk}^l\int_\sigma P_1(u)\,M_k(u,\sigma_{nk}^s)\,M_k(u,\sigma_{jk}^l)\,udu\tag{3.4}$$

$$F_2(s,p,l,k)\,R_l{}^k=-\sum_{n=1}^{2}\sum_{j=1}^{2}b_{np}^s b_{jk}^l\int_0^{a_1}\int_\sigma P_2(u)\;\times M_k(u,\sigma_{nk}^l)\sum_{j=-\infty}^{\infty}J_{m-j}(ub)[J_{j+1}(u\rho)\,q_{1j}(n,p,s,\rho)-\tag{3.5}$$

$$J_{j-1}(u\rho)\, q_{2j}(n, p, s, \rho)]\, u\rho du\, d\rho$$

$$A_l{}^k R_l{}^k = -\sum_{n=1}^{2} b_{nk}^l \int_0^{a_1} J_k(\sigma_{nk}^l \rho)\, J_k^{-1}(\sigma_{1k}^l a_1)\, \rho d\rho$$

In formulas (3.4) and (3.5) *we change to new variables of integration*, using the substitution $u = (y^2 + k^2 + l^2 + s^2)^{1/2}$. Here the radical is defined in the Riemannian plane with a slit which joins the points $\pm i\,(k^2 + l^2 + s^2)^{1/2}$, under the condition of positiveness when $y > 0$.

Let us consider system (3.3) as an operator equation in space $l_2$. The asymptotic estimates of the coefficients of the system and the substitution carried out above imply the convergence of the series

$$\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{s=1}^{\infty} |F_1(s, l, k)|^2, \quad \sum_{k=-\infty}^{\infty} \sum_{l=1}^{\infty} \sum_{p=-\infty}^{\infty} \sum_{s=1}^{\infty} |F_2(s, p, l, k)|^2$$

$$\sum_{k=-\infty}^{\infty} \sum_{l=1}^{\infty} |A_l{}^k|^2$$

Hence (3.3) generates in $l_2$ an absolutely continuous operator /6/, and the infinite system is uniquely solvable for all oscillation frequencies different from the natural frequencies of a mechanical system. The infinite system can be solved by the method of reduction, and the solutions of the abbreviated systems approach the exact solution when the order is increased.

*The properties of the contact stress singularities in the neighbourhood of the region* $\Omega_1$ are established by using the method described in /7/

$$\sigma_z^{(1)}(r, 0, \varphi) = (a_1 - r)^{-\gamma} \sigma(r, 0\, \varphi), \ \gamma = 1 - a \quad (0 < a < 1)$$

where $\sigma(r, 0, \varphi)$ is a regular function and $a$ is the root of equation

$$2\varepsilon(1 - \nu_2)(1 - \nu_1)^{-1}\left(a^2 - \sin^2 \frac{a\pi}{2}\right)\cos a\pi - \sin^2 a\pi = 0$$

and $\nu_1, \nu_2$ are Poisson's ratios.

At the boundary of the contact spot $\Omega_2$ the contact stresses have the form /1/

$$\tau_{nk}^{(2)}(r, -1) = (a_2 - r)^{-1/2}\, [r_{nk} \sin\lambda(r) + p_{nk} \cos\lambda(r)]$$
$$(r \to a_2;\ n = 1, 2)\ \ \lambda(r) = A \ln[(a_2 + r)(a_2 - r)^{-1}]$$

where $A, r_{nk}, p_{nk}$ are constants.

Setting $b = 0$ everywhere we obtain the solution for the axisymmetric initial problem. Passing to the limit as $\mu_2 \to \infty$, we obtain the solution of the problem of the vibration of a cylinder on a rigid base.

*For a numerical analysis we will select the case corresponding to* $b = a_2 = 0, \mu_1 = \mu_2 = \mu$. We obtain the axisymmetric problem of the vibration of a cylinder on an elastic layer whose lower face is fixed at a single point to a rigid base. Owing to the axial symmetry the rigid adherence at a single point does not affect the solution of the problem, i.e. we can assume that the lower face of the elastic layer is in frictionless contact with the rigid base.

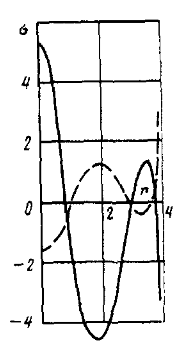


Fig. 1

This problem may be considered as a standard for comparing the effects, and enables us to separate the effects due to the presence of a region of rigid adherence $\Omega_2$. In the case considered here the infinite algebraic system (3.3) is simplified due to the absence of terms generated by the region $\Omega_2$. System (3.3) was solved using the method of reduction. The calculations were carried out for $\nu_1 = \nu_2, \rho_1 = \rho_2, h_1 = 2, w^* = 0{,}001, a_1 = 4, b_2{}^2 = 5$. Curves of dimensionless contact stresses $\sigma = \mu^{-1}\sigma_z{}^{(1)} \cdot 10^4$ are shown in Fig. 1, where the real part is represented by the solid curve and the imaginary by the dashed curve. The presence of a zone of negative stresses is explained by the fact that only the dynamic component of the problem is considered. The complete problem must take into account the static load of the cylinder which presses it into the layer, preventing the formation of a separation zone. The general solution is the sum of the static and dynamic solutions. *The solution* of the static problem is obtained from the dynamic one as $\omega \to 0$.

The author thanks V.A. Babeshko for his interest and advice.

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Translated by J.J.D.

*PMM U.S.S.R.*, Vol.47, No.5, pp.650-657, 1983
Printed in Great Britain

0021-8928/83 $10.00+0.00

UDC 539.3.534.1

# DESIGN OF CIRCULAR CYLINDRICAL SHELLS OF MINIMUM WEIGHT WITH FIXED NATURAL OSCILLATION FREQUENCIES*

A.S. BRATUS'

Approximate solutions are obtained, using asymptotic methods, of the problem of the optimum design of cylindrical shells of variable thickness, of minimum weight for fixed natural oscillation frequencies in the axisymmetric and non-axisymmetric cases. Qualitative patterns of the thickness distribution for optimum solutions are obtained and analyzed.

## 1. Basic equations.

Consider the natural oscillations of a circular cylindrical shell of variable thickness. We assume that the mean surface is specified in curvilinear coordinates $x$ and $\alpha$ in such a way that the first quadratic form has the form $R^2(dx^2+d\alpha^2)$, where $R$ is the radius of the circular cylindrical shell, $x$ varies along the generatrix, and $\alpha$ is an angular coordinate that varies in the transverse direction. We shall consider shells with straight cutoffs, that, in dimensionless variables $(x, \alpha)$, occupy the rectangular region.

$$D=\{x, \alpha: 0 \leqslant x \leqslant k,\ 0 \leqslant \alpha \leqslant \alpha_0 < 2\pi\},\ k=l/R$$

where $l$ is the shell length.

The set of equations in displacements, which determines the natural oscillations of a circular cylindrical shell of variable thickness $h(x,\alpha)$ can be expressed (e.g., /1/) in the form

$$A(h)z(x,\alpha)=\lambda hz(x,\alpha);\ A(h)=\|A_{ij}(h)\|_{i,j=1,2,3} \tag{1.1}$$

$$\lambda=\rho\frac{R^2(1-\mu^2)}{E}\omega^2,\quad A_{11}=-\frac{\partial}{\partial x}h\frac{\partial}{\partial x}-\frac{1-\mu}{2}\frac{\partial}{\partial\alpha}h\frac{\partial}{\partial\alpha}$$

$$A_{12}=-\mu\frac{\partial}{\partial x}h\frac{\partial}{\partial\alpha}-\frac{1-\mu}{2}\frac{\partial}{\partial\alpha}h\frac{\partial}{\partial x}$$

$$A_{13}(h)=\mu\frac{\partial}{\partial x}h,\quad A_{31}(h)=-\mu h\frac{\partial}{\partial x}$$

$$A_{21}(h)=-\frac{(1-\mu)}{2}\frac{\partial}{\partial x}h\frac{\partial}{\partial\alpha}-\mu\frac{\partial}{\partial\alpha}h\frac{\partial}{\partial x}$$

$$A_{22}(h)=-\frac{(1-\mu)}{2}\frac{\partial}{\partial x}h\frac{\partial}{\partial x}-\frac{\partial}{\partial\alpha}h\frac{\partial}{\partial\alpha}-\delta_0^2\left[2(1-\mu)\frac{\partial}{\partial x}h^3\frac{\partial}{\partial x}+\frac{\partial}{\partial\alpha}h^3\frac{\partial}{\partial\alpha}\right]$$

$$A_{23}(h)=\frac{\partial}{\partial\alpha}h-\delta_0^2\left[2(1-\mu)\frac{\partial}{\partial x}h^3\frac{\partial^2}{\partial x\,\partial\alpha}+\mu\frac{\partial}{\partial\alpha}h^3\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial\alpha}h^3\frac{\partial^2}{\partial\alpha^2}\right]$$

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*Prikl.Matem.Mekhan., 47,5,805-814,1983